

A combinatorial proof of the log-concavity of the numbers of permutations with k runs

Miklós Bóna *

Richard Ehrenborg †

Abstract

We combinatorially prove that the number $R(n, k)$ of permutations of length n having k runs is a log-concave sequence in k , for all n . We also give a new combinatorial proof for the log-concavity of the Eulerian numbers.

1 Introduction

Let $p = p_1p_2 \cdots p_n$ be a permutation of the set $\{1, 2, \dots, n\}$ written in the one-line notation. We say that p changes direction at position i , if either $p_{i-1} < p_i > p_{i+1}$, or $p_{i-1} > p_i < p_{i+1}$, in other words, when p_i is either a *peak* or a *valley*. We say that p has k runs if there are $k - 1$ indices i so that p changes direction at these positions. So for example, $p = 3561247$ has 3 runs as p changes direction when $i = 3$ and when $i = 4$. A geometric way to represent a permutation and its runs by a diagram is shown on Figure 1. The runs are the line segments (or edges) between two consecutive entries where p changes direction. So a permutation has k runs if it can be represented by k line segments so that the segments go “up” and “down” exactly when the entries of the permutation do. The theory of runs has been studied in [3, Section 5.1.3] in connection with sorting and searching.

In this paper, we are going to study the numbers $R(n, k)$ of permutations of length n or, in what follows, n -permutations with k runs. We will show that for any fixed n , the sequence $R(n, k)$, $k = 0, 1, \dots, n - 1$ is log-concave, that is, $R(n, k - 1) \cdot R(n, k + 1) \leq R(n, k)^2$. In particular, this implies [1, 5] that this same sequence is unimodal, that is, there exists an m so that $R(n, 1) \leq R(n, 2) \leq \cdots \leq R(n, m) \geq R(n, m + 1) \geq \cdots \geq R(n, n - 1)$. We will also show that roughly half of the roots of the generating function $R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k$ are equal to -1 , and give a combinatorial interpretation for the term which remains after one divides $R_n(x)$ by all the $(x + 1)$ factors. While doing that, we will also give a new proof of the well-known fact [2, 6] that the Eulerian numbers are log-concave.

*School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540. Supported by Trustee Ladislaus von Hoffmann, the Arcana Foundation. Email: bona@math.ias.edu.

†School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540. Supported by National Science Foundation, DMS 97-29992, and NEC Research Institute, Inc. Email: jrge@math.ias.edu.

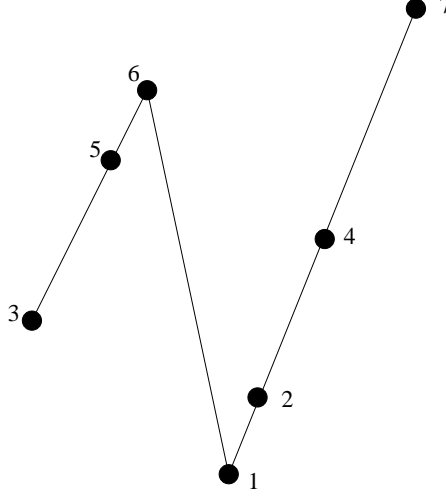


Figure 1: The permutation 3561247 has three runs

2 The Factorization of $R_n(x)$

Let $p = p_1 p_2 \cdots p_n$ be a permutation. We say that i is a *descent* of p if $p_i > p_{i+1}$, while we say that i is an *ascent* of p if $p_i < p_{i+1}$.

In our study of n -permutations with a given number of runs, we can clearly assume that 1 is an ascent of p . Indeed, taking the permutation $q = q_1 q_2 \cdots q_n$, where $q_i = n+1-p_i$, we get the *complement* of p , which has the same number of runs as p . This implies in particular that for any given i , there are as many n -permutations with k runs in which $p_i < p_{i+1}$ as there are such permutations in which $p_i > p_{i+1}$. Let $Q_n(x)$ be any generating function enumerating n -permutations according to some statistics. We say that $Q_n(x)$ is *invariant to $(i, i+1)$* if the set of n -permutations with $p_i > p_{i+1}$ contributes $Q_n(x)/2$ to Q . Certainly, in this case the set of n -permutations with $p_i < p_{i+1}$ contributes $Q_n(x)/2$ to Q as well.

Let $R_n(x) = \sum_{k=1}^{n-1} R(n, k) x^k$ be the ordinary generating function of n -permutations with k runs, where $1 \leq k \leq n-1$. So we have $R_2(x) = 2x$, $R_3(x) = 2x + 4x^2$, and $R_4(x) = 10x^3 + 12x^2 + 2x$. One sees that all coefficients of $R_n(x)$ are even, which is explained by the symmetry described above.

The following proposition is our initial step of factoring $R_n(x)$. It will lead us to the definition of an important version of this polynomial.

Proposition 2.1 *For all $n \geq 4$, the polynomial $R_n(x)$ is divisible by $(x+1)$.*

Proof: It is straightforward to verify (by considering all possible patterns for the last four entries of p) that the involution I_1 interchanging p_{n-1} and p_n increases the number of runs by 1 in half of all permutations, and, being an involution, decreases the number of runs by 1 in the other half of the permutations. In particular, there are as many permutations with an odd number of runs as there are with an even number of runs, so $(x+1)$ is indeed a divisor of $R_n(x)$. \diamond

Example 2.2 If $n = 4$, then there is 1 permutation with 1 run, 6 permutations with 2 runs and 5 permutations with 3 runs. So $R_4(x) = 2(5x^3 + 6x^2 + x) = 2(x + 1)(5x^2 + x)$.

We want to extend the result of Proposition 2.1 by proving that $R_n(x)$ has $(x + 1)$ as a factor with a large multiplicity, and also, we want to find a combinatorial interpretation for the polynomial obtained after dividing $R_n(x)$ by the highest possible power of $(x + 1)$. For that purpose, we introduce the following definition.

Definition 2.3 For $j \leq m = \lfloor (n - 2)/2 \rfloor$, we say that p is a j -half-ascending permutation if, for all positive integers $i \leq j$, we have $p_{n+1-2i} < p_{n+2-2i}$. If $j = m$, then we will simply say that p is a half-ascending permutation.

So p is a 1-half-ascending permutation if $p_{n-1} < p_n$. In a j -half-ascending permutation, we have j relations, and they involve the rightmost j disjoint pairs of entries. The term half-ascending refers to the fact that at least half of the involved positions are ascents. There are $n! \cdot 2^{-j}$ j -half-ascending permutations

Now we define a modified version of the polynomials $R_n(x)$ for j -half-ascending permutations.

Definition 2.4 Let p be a j -half-ascending permutation. Let $r_j(p)$ be the number of runs of the substring $p_1, p_2, \dots, p_{n-2j}$, and let $s_j(p)$ be the number of descents of the substring $p_{n-2j}, p_{n+1-2j}, \dots, p_n$. Denote $t_j(p) = r_j(p) + s_j(p)$, and define

$$R_{n,j}(x) = \sum_{p \in S_n} x^{t_j(p)}.$$

In particular, we will denote $R_{n,m}(x)$ by $T_n(x)$, that is, $T_n(x)$ is the generating function for half-ascending permutations.

So in other words, we count the runs in the non-half-ascending part and we count the descents in the half-ascending part (and on that part, as it will be discussed, the number of descents determines that of runs.)

Corollary 2.5 For all $n \geq 4$ we have

$$\frac{R_n(x)}{x + 1} = R_{n,1}(x).$$

Moreover, $R_{n,1}(x)$ is invariant to $(i, i + 1)$ for all $i \leq n - 3$.

Proof: Recall from proof of Proposition 2.1 that involution I_1 makes pairs of permutations, and each pair contains two elements whose numbers of runs differ by 1. Note that half of these pairs consist of two elements with $p_{n-3} < p_{n-2}$ and the other half consist of two elements with $p_{n-3} > p_{n-2}$. As $R_n(x)$ is invariant to $(n - 3, n - 4)$, it suffices to consider the first case. Dividing $R_n(x)$ by $(x + 1)$ we

obtain the run-generating function for the set of permutations which contains one element of each of these pairs, namely, the one having the smaller number of runs. Observe that for these permutations, the number of runs is equal to the value of $t_1(p)$ for the permutation in that pair in which $p_{n-1} < p_n$ (by checking both possibilities $p_{n-2} < p_{n-1}$ and $p_{n-2} > p_{n-1}$), so $R_n(x)/(x+1) = R_{n,1}(x)$. Note that our argument also proves that those permutations with $p_i < p_{i+1}$ contribute exactly $R_{n,1}(x)/2$ to $R_{n,1}(x)$, as they represent half of $R_n(x)$ divided by $(x+1)$, so our second claim is proved, too. \diamond

We point out that it is not true in general that in each pair made by I_1 , the permutation having the smaller number of runs is the one with $p_{n-1} < p_n$. What is true is that we can *suppose* that $p_{n-1} < p_n$ if we count permutations by the defined parameter $t_1(p)$ instead of the number of runs. This latter could be viewed as the $t_0(p)$ parameter.

Example 2.6 If $n = 4$, then we have 6 permutations in which $p_3 < p_4$ and $p_1 < p_2$: 1234, 1324, 1423, 2314, 2413, 3412. We have $t_1(1234) = 1$ and $t_1(p) = 2$ for all the other five permutations, showing that indeed, $R_{4,1}(x) = 2(5x^2 + x)$.

For $1 \leq j \leq m$, let I_j be the involution interchanging p_{n+1-2j} and p_{n+2-2j} . Then the following strong result generalizes Proposition 2.1.

Lemma 2.7 *For all $n \geq 4$ and $1 \leq j \leq m$, we have*

$$\frac{R_n(x)}{(x+1)^j} = R_{n,j}(x),$$

where $m = \lfloor (n-2)/2 \rfloor$. Moreover, $R_{n,j}(x)$ is invariant to $(i, i+1)$ for $i \leq n-2j-1$.

Proof: By induction on j . For $j = 1$, the statement is true by Proposition 2.1. Now suppose we know the statement for $j-1$.

To prove that $R_{n,j-1}(x)/R_{n,j}(x) = x+1$, we need to group all $(j-1)$ -half-ascending permutations in pairs, so that the t_{j-1} values of the two elements of any given pairs differ by one, and show that the set of permutations consisting of the elements of each pair having the smaller t_{j-1} value yields the generating function $R_{n,j}(x)$.

However, I_j just does that, as can be checked by verifying both possibilities $p_{n-2j} < p_{n+1-2j}$ and $p_{n-2j} > p_{n+1-2j}$. These are the only cases to consider as we can assume by our induction hypothesis that $p_{n-1-2j} < p_{n-2j}$. Moreover, permutations with $p_i < p_{i+1}$ contribute exactly $R_{n,j}(x)/2$ to $R_{n,j}(x)$ if $i \leq n-2j-1$ as they represent half of $R_{n,j-1}(x)$ divided by $(x+1)$. \diamond

Note that we have just repeated the proof of Proposition 2.1 with general j , instead of $j = 1$.

Corollary 2.8 *We have*

$$\frac{R_n(x)}{(x+1)^m} = T_n(x).$$

So we have proved that $m = \lfloor (n-2)/2 \rfloor$ of the roots of $R_n(x)$ are equal to -1 , and certainly, one other root is equal to 0 as all permutations have at least one run. It is possible to prove analytically [6] that the other half of the roots of $R_n(x)$, that is, the roots of $T_n(x)$, are all, real, negative, and distinct. That implies [5] that the coefficients of $R_n(x)$ and $T_n(x)$ are log-concave.

However, in the next section we will *combinatorially* prove that the coefficients of $T_n(x)$ form a log-concave sequence. Let $U(n, k)$ be the coefficient of x^k in $T_n(x)$. Let $\mathcal{U}(n, k)$ be the set of half-ascending permutations with k descents, so $|\mathcal{U}(n, k)| = U(n, k)$.

Now suppose for shortness that n is even and assume that p is a half-ascending permutation, that is, $p_{2i-1} < p_{2i}$ for all i , $1 \leq i \leq n/2$. The following proposition summarizes the different ways we can describe the same parameter of p .

Proposition 2.9 *Let p be a half-ascending permutation. Then p has $2k+1$ runs if and only if p has k descents, or, in other words, when $t(p) = k+1$.*

If n is odd, then the rest of our argument is a little more tedious, though conceptionally not more difficult. We do not want to break the course of our proof here, so we will go on with the assumption that n is even, then, in the second part of the proof of Theorem 4.2, we will indicate what modifications are necessary to include the case of odd n .

So in order to prove that the sequence $R(n, k)$ is log-concave in k , we need to prove that the sequence $U(n, k)$ enumerating half-ascending n -permutations with k descents is log-concave. That would be sufficient as the convolution of two log-concave sequences is log-concave [5].

3 A lattice path interpretation

Following [2], we will set up a bijection from the set $\mathcal{A}(n, k)$ of n -permutations with k descents onto that of labeled northeastern lattice paths with n edges, exactly k of which are vertical. However, our lattice paths will be different from those in [2]; in particular, they will preserve the information if the position i is an ascent or descent.

Let $\mathcal{P}(n)$ be the set of labeled northeastern lattice paths with the n edges a_1, a_2, \dots, a_n and the corresponding positive integers as labels e_1, e_2, \dots, e_n so that the following hold:

- (1) the edge a_1 is horizontal and $e_1 = 1$,
- (2) if the edges a_i and a_{i+1} are both vertical, or both horizontal, then $e_i \geq e_{i+1}$,
- (3) if a_i and a_{i+1} are perpendicular to each other, then $e_i + e_{i+1} \leq i + 1$.

We will not distinguish between paths which can be obtained from each other by translations. Let $\mathcal{P}(n, k)$ be the set of all such labeled lattice paths which has k vertical edges, and let $P(n, k) = |\mathcal{P}(n, k)|$.

Proposition 3.1 *The following two properties of paths in $\mathcal{P}(n)$ are immediate from the definitions.*

- For all $i \geq 2$, we have $e_i \leq i - 1$.
- Fix the label e_i . Then if e_{i+1} can take value v , then it can take all nonnegative integer values $w \leq v$.

Also note that all restrictions on e_{i+1} are given by e_i , independently of preceding e_j , $j < i$. The following bijection is the main result in this section.

Theorem 3.2 *The following description defines a bijection from $\mathcal{A}(n)$ onto $\mathcal{P}(n)$. Let p be a permutation on n elements. To obtain the edge a_i and the label e_i for $2 \leq i \leq n$, restrict the permutation p to the i first entries and relabel the entries to obtain the permutation $q = q_1 \cdots q_i$.*

- If the position $i - 1$ is a descent of the permutation p (equivalently, of the permutation q), let the edge a_i be vertical and the label e_i be equal to q_i .
- If the position $i - 1$ is an ascent of the permutation p , let the edge a_i be horizontal and the label e_i be $i + 1 - q_i$.

Moreover, this bijection restricts naturally to a bijection between $\mathcal{A}(n, k)$ and $\mathcal{P}(n, k)$ for $0 \leq k \leq n - 1$.

Proof: It is straightforward to see that the map described is injective on the set of labeled lattice path, not necessarily satisfying conditions (2) and (3). Assume that i and $i + 1$ are both descents of the permutation p . Let q , respective r , be the permutation when restricting to the i , respective $i + 1$, first elements. Observe that q_i is either r_i or $r_i - 1$. Since $r_i > r_{i+1}$ we have $q_i \geq r_{i+1}$ and condition (2) is satisfied in this case. By similar reasoning the three remaining cases are shown, hence the map is into the set $\mathcal{P}(n)$.

To see that this is a bijection, we show that we can recover the permutation p from its image. It is sufficient to show that we can recover p_n , and then use induction on n for the rest of p . To recover p_n from its image, simply recall that p_n is equal to the label l of the last edge if that edge is vertical, and to $n + 1 - l$ if that edge is horizontal. \diamond

The lattice path corresponding to the permutation 243165 is shown on Figure 2.

The difference between our bijection and that of [2] is that in ours, the direction of a_i tells us whether p_{i-1} is a descent in p . This is why we can use this bijection to gain information the class of half-ascending permutations.

Corollary 3.3 *The bijection in Theorem 3.2 restricts to a bijection from $\mathcal{U}(n, k)$ to lattice paths in $\mathcal{P}(n, k)$ where a_i is horizontal for all even indices i .*

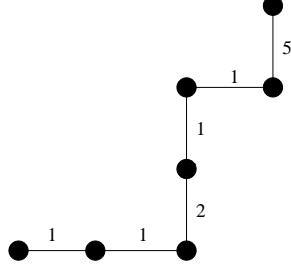


Figure 2: The image of the permutation 243165

4 The log-concavity of $U(n, k)$

In this section we are going to give a new proof for the fact that the numbers $A(n, k) = |\mathcal{A}(n, k)|$ are unimodal in k , for any fixed n . This fact is already known and has elegant proofs [2]. However, our proof will also indicate the unimodality of the $U(n, k)$.

Theorem 4.1 *For all positive integers n and all positive integers $k \leq n$ we have*

$$A(n, k-1) \cdot A(n, k+1) \leq A(n, k)^2$$

and also

$$U(n, k-1) \cdot U(n, k+1) \leq U(n, k)^2.$$

Proof: To prove the theorem combinatorially, we construct a *quasi-injection*

$$\Phi : \mathcal{P}(n, k-1) \times \mathcal{P}(n, k+1) \longrightarrow \mathcal{P}(n, k) \times \mathcal{P}(n, k).$$

By quasi-injection we mean that there will be some elements of $\mathcal{P}(n, k-1) \times \mathcal{P}(n, k+1)$ for which Φ will not be defined, but the number of these elements will be less than that of elements in $\mathcal{P}(n, k) \times \mathcal{P}(n, k)$ which are not in the image of Φ .

In particular, the restriction of Φ onto $\mathcal{V}(n, k-1) \times \mathcal{V}(n, k+1)$ will map into $\mathcal{V}(n, k) \times \mathcal{V}(n, k)$, where $\mathcal{V}(n, k)$ is the subset of $\mathcal{P}(n, k)$ consisting of lattice paths in which a_i is horizontal for all even i . Let $(P, Q) \in \mathcal{P}(n, k-1) \times \mathcal{P}(n, k+1)$. Place the initial points of P and Q at $(0, 0)$ and $(1, -1)$, respectively. Then the endpoints of P and Q are $(n-k+1, k-1)$ and $(n-k, k)$, respectively, so P and Q intersect. Let X be their *first* intersection point (we order intersection points from southwest to northeast), and decompose $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, where P_1 is a path from $(0, 0)$ to X , P_2 is a path from X to $(n-k, k)$, Q_1 is a path from $(1, -1)$ to X , and Q_2 is a path from X to $(n-k+1, k-1)$. Let $P' = P_1 \cup Q_2$ and let $Q' = Q_1 \cup P_2$. If P' and Q' are valid paths, that is, if their labeling fulfills conditions (1)–(3), then we set $\Phi(P, Q) = (P', Q')$. See Figure 3 for this construction.

It is clear that $\Phi(P, Q) = (P', Q') \in \mathcal{P}(n, k) \times \mathcal{P}(n, k)$, (in particular, (P', Q') belongs to the subset of $\mathcal{P}(n, k) \times \mathcal{P}(n, k)$ consisting of *intersecting* pairs of paths), and that Φ is one-to-one. What remains to show is that the number of pairs $(P, Q) \in \mathcal{P}(n, k-1) \times \mathcal{P}(n, k+1)$ for which Φ cannot be defined this way is less than the number of pairs $(P', Q') \in \mathcal{P}(n, k) \times \mathcal{P}(n, k)$ which are not obtained as images of Φ .



Figure 3: Constructing the new pair of paths

In fact, we will show that this is true even if we restrict ourselves to pairs $(P', Q') \in \mathcal{P}(n, k) \times \mathcal{P}(n, k)$ which do intersect.

Let a, b, c, d be the labels of the four edges adjacent to X as shown in Figure 4, the edges AX and XB originally belonging to P and the edges CX and XD originally belonging to Q . (It is possible that these four edges are not all distinct; A and C are always distinct as X is the first intersection point, but it could be, that $B = D$ and so $BX = DX$; this singular case can be treated very similarly to the generic case we describe below and hence omitted). Then a configuration shown on Figure 4 can be part of a pair (P, Q) in the domain of Φ exactly when $a \geq b$ and $c \geq d$. On the other hand, such a configuration can be part of a pair of paths (P', Q') in the image of Φ exactly when $a + d \leq i$ and $b + c \leq i$, where $i - 1$ is the sum of the two coordinates of X . Let us keep b fixed, and see what that means for a and c . The value of a can be $b, b + 1, \dots, i - 1$, so a can take $i - b$ different values, whereas the value of c can be $1, 2, \dots, i - b$ which is again $i - b$ different possibilities. Note in particular that the second set of values can be obtained from the first by simply subtracting each value from i . Then the set of all labeled paths from $(0, 0)$ to A is identical to that of paths from $(1, -1)$ to C . In particular, the distributions of the labels of the edges ending in A , respectively C , are identical, even if we also require that they end in a horizontal, or in a vertical edge.

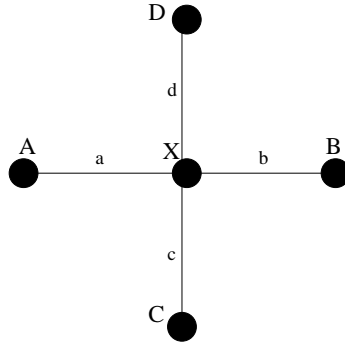


Figure 4: Labels around the point X

Let $H(X)$ be the set of all pairs of labeled paths $((0, 0), X) \times ((1, -1), X)$. Now it is easy to see that if any labeled path G from $(0, 0)$ to A allows a to be in the interval $b, b + 1, \dots, i - 1$, then the path from $(1, -1)$ to C identical to G allows c to be in the interval $1, 2, \dots, i - b$. Indeed, the edge preceding AX is either horizontal, and then it must have a label between b and $i - 2$, or it is vertical,

and then it must be between 1 and $i - b$ to make it possible for a to be in the interval $b, b + 1, \dots, i - 1$. Similarly, if the edge preceding CX is horizontal, and it has a label between b and $i - 2$, or if it is vertical, and has a label between 1 and $i - b$, then it makes it possible for c to be in the interval $1, 2, \dots, i - b$. (And certainly, if the edge preceding CX is horizontal, and it has a label smaller than b , that is good, too). As the distributions of the labels of the edges ending in A , respectively C are identical, this implies that for any fixed values of b , there are at least as many pairs of paths in $H(X)$ so that $b + c \leq i$ as there are pairs of paths in $H(X)$ with $a \geq b$. (Recall that $i - 1$ is the sum of the coordinates of X). In other words, if the pair $(\alpha, \beta) \in ((0, 0), A) \times ((1, -1), C)$ allows $a \geq b$, then the pair $(\beta, \alpha) \in ((0, 0), A) \times ((1, -1), C)$ allows $b + c \leq i$, so we can flip α and β . We point out that this is intuitively not surprising: a has to be *at least* a certain value, while c has to be *at most* a certain value, and it is clear that this second requirement is easier in our labeling.

By symmetry, if we fix d instead of b , the same holds: the number of pairs of paths in $H(X)$ so that $a + d \leq i$ is at least as large as that of pairs of paths in $H(X)$ with $c \geq d$, and that can be seen again by flipping α and β .

Finally, this same argument certainly applies if we want both conditions to be satisfied: if the pair $(\alpha, \beta) \in ((0, 0), A) \times ((1, -1), C)$ allows $a \geq b$ and $c \geq d$, then the pair $(\beta, \alpha) \in ((0, 0), A) \times ((1, -1), C)$ allows $b + c \leq i$ and $a + d \leq i$. And this is what we wanted to prove: there are at least as many pairs of paths in $\mathcal{P}(n, k) \times \mathcal{P}(n, k)$ which are not images of Φ as there are pairs of paths in $\mathcal{P}(n, k - 1) \times \mathcal{P}(n, k + 1)$ for which Φ is not defined. As Φ is one-to-one, this proves that $A(n, k - 1) \cdot A(n, k + 1) \leq A(n, k)^2$, so the sequence $\{A(n, k)\}_k$ is log-concave for all n .

To prove that the sequence $\{U(n, k)\}$ is log-concave, recall that half-ascending permutations in $\mathcal{U}(n, k)$ correspond to elements of $\mathcal{V}(n, k)$, that is, elements of $\mathcal{P}(n, k)$ in which all edges a_i are horizontal if i is even. We point out that this implies $B = D$. Then note that Φ does not change the indices of the edges, in other words, if $\Phi(P, Q) = (P', Q')$, and a given edge northeast from X was the i th edge of path P , then it will be the i th edge of path Q' . Therefore, Φ preserves the property that all even-indexed edges are horizontal, so the restriction of Φ into $\mathcal{V}(n, k - 1) \times \mathcal{V}(n, k + 1)$ maps into $\mathcal{V}(n, k) \times \mathcal{V}(n, k)$. Finally, we need to show that there are more pairs of paths in $\mathcal{V}(n, k) \times \mathcal{V}(n, k)$ which are not images of Φ than there are pairs of paths in $\mathcal{V}(n, k - 1) \times \mathcal{V}(n, k + 1)$ for which Φ is not defined. Note that the corresponding fact in the general case was a direct consequence of the fact that for any labeled path $((0, 0), A)$ was identical to a unique labeled path $((1, -1), C)$, and therefore the distributions of the labels a and c were identical. This remains certainly true if we restrict ourselves to paths in which all edges with even indices are horizontal. As any restriction of Φ is certainly one-to-one, this proves that $U(n, k - 1) \cdot U(n, k + 1) \leq U(n, k)^2$. \diamond

Now we are in a position to prove the main result of this paper.

Theorem 4.2 *The polynomial $R_n(x)$ has log-concave coefficients, for all positive integers n .*

Proof: First suppose that n is even. For $n \leq 3$, the statement is true. If $n \geq 4$, then Lemma 2.7 shows that $R_n(x) = (x + 1)^m T_n(x)$. The coefficients of $(x + 1)^m$ are just the binomial coefficients, which are certainly log-concave [4], while the coefficients of $T_n(x)$ are the $U(n, k)$, which are log-concave by Theorem 4.1 and the remark thereafter. As the product of two polynomials with log-concave coefficients has log-concave coefficients [5], the proof is complete for n even.

If n is odd, then the equivalent of Proposition 2.9 is a bit more cumbersome. Again, we let us make use of symmetry by taking complements, but instead of assuming $p_1 < p_2$, let us assume that $p_2 < p_3$. Taking $R_{n,m}(x)$ then adds the restrictions $p_4 < p_5$, $p_6 < p_7$, \dots , $p_{n-1} < p_n$. Then it is straightforward from the definition of $t_m(p)$ that $t_m(p) = d(p)$ where $d(p)$ is the number of descents of p , and we say, for shortness, that the singleton p_1 has 0 runs.

So for odd n we have $T_n^{odd}(x) = 2 \cdot \sum_{\substack{p \in S_n \\ p_2 < p_3}} x^{t_m(p)} = 2 \cdot \sum_{\substack{p \in S_n \\ p_2 < p_3}} x^{d(p)}$, and then, in order to see that the coefficients of $T_n^{odd}(x)$ are log-concave, we can repeat the argument of Theorem 4.1. Indeed, the coefficient of x^k in $T_n^{odd}(x)$ equals the cardinality of $\mathcal{V}'(n, k)$, the subset of $\mathcal{P}(n, k)$ in which the edges a_3, a_5, \dots, a_7 are horizontal. And the fact that the $|\mathcal{V}'(n, k)|$ are log-concave can be proved exactly as the corresponding statement for the $|\mathcal{V}'(n, k)| = U(n, k)$, that is, by taking the relevant restriction of Φ .

This completes the proof of the theorem for all n . \diamond

References

- [1] F. BRENTI, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, in Jerusalem combinatorics '93, *Contemp. Math.*, **178** (1994), 71–89, Amer. Math. Soc., Providence.
- [2] V. GASHAROV, On the Neggers-Stanley conjecture and the Eulerian polynomials, *J. Combin. Theory Ser. A* **82** (1998), 134–146.
- [3] D. E. KNUTH, “The art of computer programming. Volume 3,” Addison-Wesley Publishing Co., Reading, 1973.
- [4] B. E. SAGAN, Inductive and injective proofs of log concavity results, *Discrete Math.* **68** (1988), 281–292.
- [5] R. STANLEY Log-concave and unimodal sequences in algebra, combinatorics, and geometry in Graph theory and its applications: East and West (Jinan, 1986), pp. 500–535, Ann. New York Acad. Sci., 576, New York, 1989.
- [6] H. WILF, Real Zeroes of polynomials that count runs and descending runs, personal communication.